DEGENERATE SOLUTIONS OF ELECTRON-BEAM EQUATIONS

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Zhurnal Prikladnoi Mekhaniki i Teknicheskoi Fiziki, Vol. 9, No. 1,
pp. 3–12, 1968

In this paper, exact solutions are constructed for stationary electron beams that are degenerate in the Cartesian (x, y, z), axisymmetric (r, θ, z), and spiral (in the planes y = const (u, y, v)) coordinate systems. The degeneracy is determined by the fact that at least two coordinates in such a solution are cyclic or are integrals of motion. Mainly, rotational* beams are considered. Invariant solutions for beams in which the presence of vorticity resulted in a linear dependence of the electric-field potential φ on the above coordinates were considered in [1]. In degenerate solutions, the presence of vorticity results in a quadratic or more complex dependence of the potential on the coordinates that are integrals of motion. In [2]** and in a number of papers referred to in [2], the degenerate states of irrotational beams are described. The known degenerate solutions for rotational beams apply to an axisymmetric one-dimensional (r) beam with an azimuthal velocity component [3] and to relativistic conical flow [1]. The equations used below follow from the system of electron hydrodynamic equations for a stationary relativistic beam

$$\sum_{\beta=1}^{3}\frac{\partial}{\partial q^{\beta}}\left[\sqrt{\gamma}g^{\beta\beta}g^{\alpha\alpha}\left(\frac{\partial A_{\alpha}}{\partial q^{\beta}}-\frac{\partial A_{\beta}}{\partial q^{\alpha}}\right)\right]=4\pi\rho\sqrt{\gamma}g^{\alpha\alpha}u_{\alpha},$$

$$\sum_{\beta=1}^{3}\frac{\partial}{\partial q^{\beta}}\left(\sqrt{\gamma}g^{\beta\beta}\frac{\partial\varphi}{\partial q^{\beta}}\right)=4\pi\rho\sqrt{\gamma}u,\qquad\sum_{\beta=1}^{3}g^{\beta\beta}u_{\beta}^{2}+1=u^{2}$$

$$\frac{\eta}{c}u\frac{\partial\mathscr{E}}{\partial q^{\alpha}}=\sum_{\beta=1}^{3}g^{\beta\beta}u_{\beta}\left(\frac{\partial p_{\beta}}{\partial q^{\alpha}}-\frac{\partial p_{\alpha}}{\partial q^{\beta}}\right),$$

$$\sum_{\beta=1}^{3}\frac{\partial}{\partial q^{\beta}}\left(\sqrt{\gamma}g^{\beta\beta}\rho u_{\beta}\right)=0,\quad u\equiv\frac{\eta}{c^{2}}(\varphi+\mathscr{E})+1,$$

$$cu_{\alpha}\equiv\frac{\eta}{c}A_{\alpha}+p_{\alpha},\quad\alpha,\beta=1,2,3,\quad\gamma\equiv g_{11}g_{22}g_{33}$$

where q^{β} denotes orthogonal coordinates with the metric tensor $g^{\beta\beta}$ (β = 1, 2, 3); A_{α} is the magnetic potential; $V_{\alpha} = (u_{\alpha}/u)c$ is the electron velocity; ρ is the scalar space-charge density ($\rho > 0$); is the energy in eV; p_{α} is the generalized momentum of an electron per unit mass; η is the electron charge-mass ratio.

§1. Solenoidal beams. Solutions are constructed below for plane and axisymmetric rotational beams, all of whose parameters are integrals of motion.

1.1. An axisymmetric and one-dimensional (r) beam with four velocity components (0, u_{θ}, u_z, and $-u$) is described by the equations

$$\frac{r}{u_{\theta}}\frac{d}{dr}\frac{1}{r}\frac{dA_{\theta}}{dr}=\frac{1}{u_{z}r}\frac{d}{dr}r\frac{dA_{z}}{dr}=\frac{1}{ur}\frac{d}{dr}r\frac{d\varphi}{dr}=4\pi\rho,$$

$$u^{2}=\frac{1}{r^{2}}u_{\theta}^{2}+u_{z}^{2}+1,\ \frac{\eta}{c}u\frac{d\mathscr{E}}{dr}=\frac{1}{r^{2}}u_{\theta}\frac{dp_{\theta}}{dr}+u_{z}\frac{dp_{z}}{dr},\tag{1.1}$$

$$cu_{\theta}\equiv\frac{\eta}{c}A_{\theta}+p_{\theta},\quad cu_{z}\equiv\frac{\eta}{c}A+p_{z},$$

$$u\equiv\frac{\eta}{c^{2}}(\varphi+\mathscr{E})+1.$$

System (1.1) imposes only one condition on the three arbitrary functions $\mathscr{E}(r)$, $p_{\theta}(r)$, and $p_z(r)$. This indeterminacy is removed if the specific method of beam formation is taken into account.

Let beam (1.1) be formed under axisymmetric and stationary conditions. Then $\mathscr{E}$ and p_{θ} are integrals of motion and are defined in terms of the field potentials on the cathode surface $r = r_k(z)$:

$$\mathscr{E}=-\varphi_{k}(r_{k}),\quad p_{\theta}=-\frac{\eta}{c}A_{\theta k}(r_{k}).\tag{1.2}$$

The electron current along z, which is bounded by the trajectory tube

$$c\int_{0}^{r_k}\rho u_{z}2\pi r\,dr=J(r_{k}),\tag{1.3}$$

is also an integral of motion. The setting of these quantities, which are determined by characteristics of the gun, adds two deficient conditions: $\mathscr{E} = \mathscr{E}(J)$ and $p_{\theta} = p_{\theta}(J)$. On the other hand, the problem can be formulated actively [4]: namely, determine the conditions of beam formation by assigning the motion parameters.

Let, for example, the cathode be equipotential, $\varphi_k = -\mathscr{E} = 0$, and let it be required to calculate the beam in a homogeneous magnetic field H in the absence of rotation $u_{\theta} = 0$. From (1.1) follows the solution

$$u=\operatorname{ch}\lambda\sigma,\quad u_{z}=\operatorname{sh}\lambda\sigma,\quad A_{\theta}=\frac{1}{2}Hr^{2},\quad\rho=\frac{c^{2}}{4\pi\eta}\frac{\lambda^{2}}{r^{2}},$$

$$p_{z}=0,\quad\sigma=\ln(r/r_{0}),\quad r_{0}=\text{const},\quad\lambda=\text{const}.$$

Hence, we obtain the formation conditions

$$p_{\theta}=-\frac{1}{2}\frac{\eta}{c}Hr^{2},\quad J=\frac{c^{3}}{2\eta}\lambda(\operatorname{ch}\lambda\sigma-1)$$

where r_0 is the characteristic radius of the beam.

Let us assume that when $\mathscr{E} = 0$ and $H_{\theta} \neq 0$ a nonrelativistic beam must have a uniform axial velocity $V_z = V$ = const. Then it is easy to obtain

$$p_{z}=-A_{0}\ln\frac{r}{r_{0}},$$

$$V_{\theta}=\frac{VA_{0}}{\Omega}\ln\left(1+\frac{\Omega V_{\theta}}{VA_{0}}\right)+\frac{\Omega}{2}(r^{2}-r_{1}^{2}),$$

$$p_{\theta}=V_{\theta}-\frac{\Omega}{2}r^{2},\tag{1.4}$$

$$\eta\frac{d\varphi}{dr}=\frac{2\eta}{r}\left(\frac{J}{V}-Q\right)=\frac{VA_{0}}{r}+\frac{\Omega^{2}}{4}r-\frac{1}{r^{3}}p_{\theta}^{2},$$

$$A_{0}=\text{const},\quad\frac{\eta}{c}H_{\theta}\equiv A_{0},\quad\frac{\eta}{c}H_{z}=\Omega,$$

*Here and below, a rotational beam is a beam with a rotational field of generalized momentum.

**Note that the second example in §6 of [2] is incorrect.

where r_0 and r_1 are characteristic radii; 0, H_θ, and H_z are the components of the external magnetic field; Q is the charge on an internal rod that can be situated in the field of the beam.

1.2. The system of equations of a plane, one-dimensional (x) beam with four-velocity (0, u_y, u_z, and $-u$) has the form

$$\frac{A_y''}{u_y} = \frac{A_z''}{u_z} = \frac{\varphi''}{u} = 4\pi\rho,$$

$$\frac{\eta}{c} u\mathscr{E}' = u_y p_y' + u_z p_z', \quad \left(\varphi' \equiv \frac{d\varphi}{dx}\right), \tag{1.5}$$

$$u^2 = u_y^2 + u_z^2 + 1,$$

$$cu_y = \eta c^{-1} A_y + p_y, \quad cu_z = \eta c^{-1} A_z + p_z.$$

System (1.5) admits of the integral

$$(\eta c^{-2})^2 [(\varphi')^2 - (A_y')^2 - (A_z')^2] = -k^2 = \text{const}. \tag{1.6}$$

Let the beam at $\mathscr{E} = 0$ have a uniform velocity along the y-axis: $u_y = \beta u$, β = const. In this case,

$$u = (1-\beta^2)^{-1/2} \operatorname{ch}\psi, \; u_z = \operatorname{sh}\psi,$$

$$4\pi\, \eta\, c^{-2}\, \rho = (\psi')^2 + \psi'' \operatorname{th}\psi,$$

$$p_y = (c/a)x, \; p_z' = -(c/a)\, \beta\, (1-\beta^2)^{-1/2} \operatorname{cth}\psi,$$

$$a\,\psi' \operatorname{sh}\psi = \beta\,(1-\beta^2)^{-1/2}\,(\alpha \operatorname{sh}\psi - 1),$$

where a and α are arbitrary constants. Integration of the last equation gives

$$\frac{\alpha\beta}{\sqrt{1-\beta^2}} \frac{x}{a} = \psi - \frac{2}{\sqrt{1+\alpha^2}} \operatorname{Ar\,th} \frac{\alpha + \operatorname{th} {}^1/_2 \psi}{\sqrt{1+\alpha^2}},$$

$$J = c\int_0^x \rho u_z dx = \frac{c^3\beta\alpha \operatorname{ch}\psi}{4\pi\eta a \sqrt{1-\beta^2}}, \quad \frac{4\pi\eta}{c^2}\rho = \frac{\psi'\beta\alpha}{a\sqrt{1-\beta^2}}.$$

A beam with a uniform space-charge density is obtained if

$$\eta\, c^{-2}\, \mathscr{E} = (1-\alpha)\, u, \quad p_y = (1-\alpha)\, cu_y,$$

$$p_z = (1-\alpha)\, c\, u_z,$$

then

$$(4\pi\eta/c^2)\,\rho = \alpha k^2, \quad u = B \operatorname{ch} kx,$$

$$u_y = B_y \operatorname{sh} kx, \quad u_z = B_z \operatorname{sh} kx,$$

where B, B_y, and B_z are constants, and $B^2 = B_y^2 + B_z^2$. As distinct from an irrotational beam [2], the charge density is increased by a factor of α.

1.3. A plane two-dimensional (x,y) beam with four-velocity components (0, $u_y(x, z)$, 0, and $-u$) satisfies the equations

$$u^2 = u_y^2 + 1, \quad (\eta/c)\, ud\mathscr{E} = u_y dp_y,$$

$$u_y = (\eta/c)\, A_y + p_y, \tag{1.7}$$

$$\frac{1}{u_y}\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A_y}{\partial z^2}\right) = \frac{1}{u}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2}\right) = 4\pi\rho.$$

For $p_y = \; = 0$, the solution is known [5]:

$$u = \operatorname{ch} f, \quad u_y = \operatorname{sh} f, \quad (4\pi\eta/c^2)\,\rho = (\nabla f)^2, \tag{1.8}$$

$$\nabla^2 f = 0, \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \quad \nabla \equiv \left(\frac{\partial}{\partial x}, 0, \frac{\partial}{\partial z}\right).$$

If we let $u = \operatorname{ch}\psi$ and $u_y = \operatorname{sh}\psi$ in (1.7), we obtain

$$\frac{d^2\psi}{df^2}\left(c \operatorname{ch}\psi - \frac{dp_y}{d\psi}\right) = \left(\frac{d\psi}{df}\right)^2 \left(\frac{d^2 p_y}{d\psi^2} - \operatorname{th}\psi \frac{dp_y}{d\psi}\right),$$

$$\frac{\eta}{c}\frac{d\mathscr{E}}{d\psi} = \operatorname{th}\psi \frac{dp_y}{d\psi},$$

from which it follows that for any dependence $\psi(f)$

$$\frac{1}{c} p_y = \operatorname{sh}\psi - \alpha\int \operatorname{ch}\psi\, df, \quad \frac{\eta}{c^2}\mathscr{E} = \operatorname{ch}\psi - \alpha\int \operatorname{sh}\psi df,$$

$$\frac{4\pi\eta}{c^2}\rho = \alpha(\nabla f)^2 \frac{d\psi}{df}, \quad \frac{\eta}{c^2} A_y = \alpha\int \operatorname{ch}\psi\, df,$$

$$\frac{\eta\varphi}{c^2} = \alpha\int \operatorname{sh}\psi\, df - 1, \tag{1.9}$$

where α is an arbitrary constant, In particular, when $\psi = f$, solution (1.9) differs from irrotational solution (1.8) by the coefficient α.

1.4. An axisymmetric two-dimensional (r, z) beam with (0, u_θ, 0, and $-u$) is defined by the equations

$$\frac{r}{u_\theta}\left(\Delta - \frac{1}{r^2}\right)\frac{A_\theta}{r} = \frac{\Delta\varphi}{u} = 4\pi\rho, \tag{1.10}$$

$$\Delta \equiv \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

$$u = \frac{1}{r^2}\, u_\theta^2 + 1, \quad \frac{\eta}{c} ud\mathscr{E} = \frac{u_\theta}{r^2} dp_\theta,$$

$$cu_\theta = \frac{\eta}{c} A_\theta + p_\theta.$$

The relation $p_\theta = \; \eta/\omega$, where ω = const is the angular velocity, leads to the case of uniform beam rotation

$$u = (1-\beta^2)^{-1/2}, \quad u_\theta = r\beta\,(1-\beta^2)^{-1/2}, \quad \beta = r\omega/c.$$

Here, system (1.10) reduces to a linear equation for $\mathscr{E}'$

$$\mathscr{E}' \equiv \mathscr{E} - \frac{2c^2}{\eta\sqrt{1-\beta^2}}, \tag{1.11}$$

$$\frac{\partial^2\mathscr{E}'}{\partial r^2} - \frac{1+\beta^2}{1-\beta^2}\frac{1}{r}\frac{\partial\mathscr{E}'}{\partial r} + \frac{\partial^2\mathscr{E}'}{\partial z^2} = 0.$$

At the nonrelativistic limit, it follows from (1.10) that

$$\left(\Delta - \frac{1}{r^2}\right)\frac{A_\theta}{r^2} = 0,$$

$$\eta\frac{d\mathscr{E}}{dp_\theta} = \frac{V_\theta}{r^2}, \quad \eta(\varphi + \mathscr{E}) = \frac{V_0^2}{2r^2}, \quad 4\pi\rho = \Delta\varphi$$

If a certain relation $p_\theta = p_\theta(F)$ is given, the first two equations reduce to one for F. Let $p_\theta = r_0^2 F$, where r_0 = const; then

$$2\eta\mathscr{E} = r_0^2 F^2, \quad 2\eta\varphi = (r^2 - r_0^2)\, F^2,$$

$$\Delta F + \frac{2}{r}\frac{r^2 + r_0^2}{r^2 - r_0^2}\frac{\partial F}{\partial r} = 0.$$

i.e., the problem reduces to a linear equation for F. When $F = \omega$ = const, we obtain a nonrelativistic equivalent of the example examined above:

$$p_\theta = \frac{\eta}{\omega}\mathscr{E}, \quad 2\eta(\varphi + \mathscr{E}) = \omega^2 r^2, \quad 4\pi\eta\rho = 2\omega^2 - 2\frac{\eta}{r}\frac{\partial\mathscr{E}}{\partial r}.$$

The function $\mathscr{E}$ must be found from (1.11) as $\beta \to 0$.

§ 2. A relativistic plane beam. Let the beam have a two-valued velocity of the form $u_z = \pm w(z)$, i. e., let it consist of two subcurrents that move in opposition along the z-axis. Let the subcurrent densities equal $\rho/2$ and give a total charge density in the beam of ρ

$$\frac{\partial}{\partial z}\rho w = 0, \quad \rho = I/cw, \quad I = \text{const}. \tag{2.1}$$

In this case, there is no beam current along the z-axis, but there is a so-called rotational current of I/2.

Solutions are constructed below for beams with density (2.1), in which the electron current is directed along the cyclic coordinate. These beams can be interpreted as single-flow beams with the current I along the z-axis only when the limitation on beam width is sufficient to make the magnetic field of the current I negligible.

2.1. A plane irrotational beam with four-velocity components

$$u_x(z),\ u_y = a(z)\operatorname{sh}[kx + \psi(z)],\ u_z = \pm w,\ u = a \operatorname{ch}[kx + \psi],$$

satisfies the equations

$$\frac{1}{u_x}\frac{\partial^2 u_x}{\partial z^2} = \frac{\nabla^2 u_y}{u_y} = \frac{\nabla^2 u}{u} = \frac{4\pi\eta I}{c^3 w}, \quad w^2 = a^2 - u_x^2 + 1, \tag{2.2}$$

where ∇^2 is defined by (1.8). Then the problem reduces to a one-dimensional equation for a, u_x, and ψ

$$\frac{u_x''}{u_x} = \frac{a''}{a} + k^2 + (\psi')^2 = \frac{\lambda^2}{2w}, \quad 2\frac{a'}{a} = -\frac{\psi''}{\psi'},$$
$$\lambda^2 \equiv \frac{8\pi\eta}{c^3} I \quad \left(a' \equiv \frac{da}{dz}\right). \tag{2.3}$$

Equations (2.3) admit the integrals

$$\psi' = a^{-2} d,\ (a')^2 + k^2 a^2 - a^{-2} d^2 =$$
$$= \lambda^2 w + b \quad (b,\ d = \text{const}). \tag{2.4}$$

The case of k = 0 is considered in [6]. Let $u_x = 0$ and $k \neq 0$. Then

$$\frac{1}{\lambda^2}\frac{w^2 (w')^2}{1 + w^2} = b - U,$$
$$U \equiv \frac{-w}{1 + w^2}(1 + w^2 - m^2 w - n^2 w^3). \tag{2.5}$$
$$m^2 = (k^2 + d^2)\lambda^{-2}, \quad n^2 = k^2 \lambda^{-2}.$$

Following [7], we can say that Eq. (2.5) describes the motion a fictitious particle with the energy b in a field with the potential U. From Fig. 1 we can determine the reversal point of the particle w_m for various b. With a uniform space charge (b = 0), there are two reversal points: w = 0 and $w = w_m$. In this case, the motion is periodic and can be represented as the waves w along the z-axis. For the wavelength L (Fig. 2)

$$\lambda L = 2\int_0^{w_m} \frac{\sqrt{w}\,dw}{\sqrt{1 + w^2 - m^2 w - n^2 w^3}}.$$

At the nonrelativistic limit, this solution becomes the known solution of [8], which describes that wavy perturbation of the potential along the z-axis which is superposed on a plane Brillouin beam.

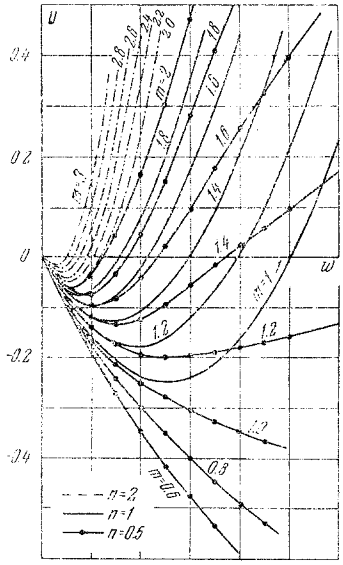

Fig. 1

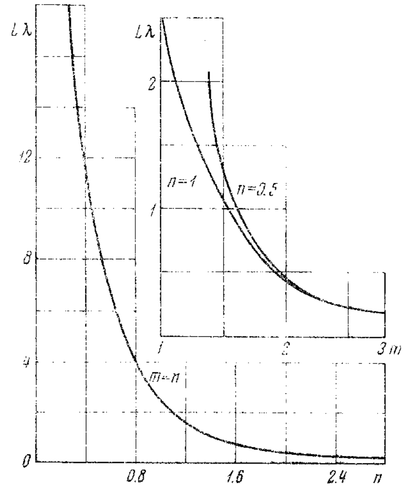

Fig. 2

2.2. The solution in the preceding paragraph can be extended to a rotational beam with four-velocity components

$$0,\ u_y = a(z)\ \operatorname{sh}\ kx,$$
$$u_z = \pm w(z), \quad u = a(z)\ \operatorname{ch} kx. \tag{2.6}$$

If we take the momentum p_y and the energy in the form

$$p_y = cp\ \operatorname{sh} kx, \quad \eta\mathscr{E} = c^2 p\ \operatorname{ch} kx,$$
$$p = \text{const}, \quad k = \text{const}, \tag{2.7}$$

the equation $(\eta/c)u d\mathscr{E} = u_y dp_y$ is satisfied, and it remains to solve the equations for the field:

$$\frac{\Delta^2 A_y}{u_y} = \frac{\Delta^2\varphi}{u} = \frac{4\pi I}{cw}, \quad cu_y = \frac{\eta}{c}A_y + p_y,$$
$$u = \frac{\eta}{c^2}(\varphi + \mathscr{E}) + 1. \tag{2.8}$$

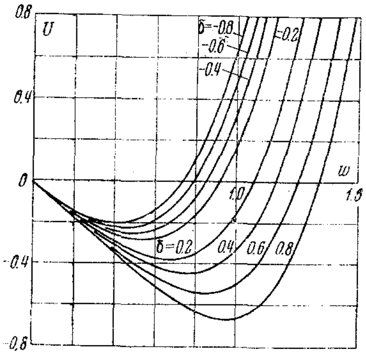

Fig. 3

Substitution of (2.6) and (2.7) into (2.8) gives

$$a'' + k^2(a - p) = \frac{\lambda^2}{2w} a, \quad a^2 - 1 = w^2, \quad \lambda \equiv \frac{8\pi\eta}{c^3} I.$$

As a result, we obtain an equation for w:

$$\frac{1}{\lambda^2} \frac{w^2 (w')^2}{1 + w^2} = b - U,$$
$$U \equiv -[w - n^2 w^2 + \delta(\sqrt{1 + w^2} - 1)] \quad w' \equiv dw/dz, \tag{2.9}$$
$$b = \text{const}, \quad n = k/\lambda, \quad \delta = pk^2\lambda^{-2}.$$

Figure 3 gives graphs of U(w), from which we can find the reversal point w_m when $U(w_m) = b$.

In the case of total space charge (b = 0), the equation describes the periodic waves w(z) with amplitude w_m (Fig. 4) and the length L:

$$\lambda L = 2 \int_0^{w_m} \frac{w dw}{\sqrt{1 + w^2} \sqrt{w - n^2 w^2 + \delta(\sqrt{1 + w^2} - 1)}}.$$

At the nonrelativistic limit, we have

$$\pm \lambda z = \alpha^{-3/2} [\arcsin \sqrt{\alpha w} - \sqrt{\alpha w (1 - \alpha w)}],$$
$$\alpha \equiv n^2 - 1/2\, \delta.$$

It is apparent that with a rather intense vorticity $\delta > 2n^2$, the solution becomes aperiodic.

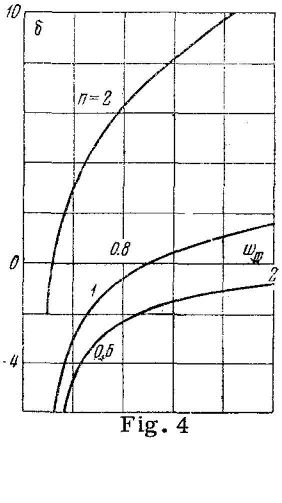

Fig. 4

§ 3. **An axial nonrelativistic beam.** Solutions for a nonplane beam that are similar to those in § 2 can be constructed only in the nonrelativistic case.

3.1. The equations for an axisymmetric beam have the form

$$V_z^2 + \frac{V_\theta^2}{r^2} = 2\eta(\varphi + \mathscr{E}), \quad \eta d\mathscr{E} = \frac{V_\theta}{r^2} dp_\theta,$$
$$V_\theta = \frac{\Omega}{2} r^2 + p_\theta, \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} = 4\pi\rho, \tag{3.1}$$
$$\rho = \frac{I}{V_z}, \quad \Omega = \frac{\eta}{c} H,$$

where H is a uniform axial magnetic field. These equations are satisfied with the assumption that $V_z = V(z)$ and $V_\theta = V_\theta(r)$, if we let

$$\varphi = \pi \rho_0 r^2 + 2Q \ln(r/r_0) + {}^1/_2\, V^2 \eta^{-1},$$
$$\rho_0 = \text{const}, \quad Q = \text{const}, \tag{3.2}$$

As a result we obtain an equation for V

$$(V\, dV/dz)^2 = 4\pi\eta(IV - {}^1/_2 \rho_0 V^2) + b,$$
$$b = \text{const}.$$

In the case of b = 0, two types of solutions are obtained:

$$z = \frac{4\pi\eta}{\omega^3} I(\omega t - \sin \omega t), \quad s = 1, \tag{3.3}$$
$$z = -\frac{4\pi\eta}{\omega^3} I(\omega t - \operatorname{sh} \omega t), \quad s = -1,$$
$$s = \operatorname{sign} \rho_0, \quad \omega^2 = |4\pi\eta\rho_0|, \quad dt = V^{-1} dz.$$

Solution (3.3) coincides with that obtained from the approximate paraxial equation in [9]. However, the precise conditions for realization of the beam in question differ from the approximate ones in [9]. From (3.1) and (3.2), we have expressions for   and p_θ:

$$2\eta \mathscr{E} = r^{-2} ({}^1/_2\, \Omega r^2 + p_\theta)^2 -$$
$$- {}^1/_2\, s\omega^2 r^2 - 4\eta Q \ln r/r_0 \tag{3.4}$$
$$p_\theta = \pm [{}^1/_4 (\Omega^2 - 2s\omega^2) r^4 - 2\eta Q r^2]^{1/2} \tag{3.5}$$

In particular, if there is no core inside the beam, Q = 0 and

$$\frac{\eta}{c} A_k = \frac{\mp J}{4\pi I} \sqrt{\Omega^2 - 2s\omega^2},$$

$$\eta\varphi_k = \frac{-J}{4\pi I} [\Omega^2 \pm \sqrt{\Omega^2 - 2s\omega^2} - 2s\omega^2],$$

where J is the electron current,* which is determined by (1.3).

3.2. The solution for an irrotational axisymmetric beam of the form of (3.3) can be extended to an elliptic beam [8]. When a vortex is introduced into such a

* The nonrelativistic beams considered in § 3 and 4 are interpreted here as single-flow beams.

beam, the solution s = −1 can be realized. Let, for example,

$$p_x = \nu_{11} x + \nu_{12} y,\ p_y = \nu_{21} x + \nu_{22} y,$$
$$2\eta\mathcal{E} = \alpha_{11} x^2 + 2\alpha_{12} x y + \alpha_{22} y^2,$$

where $\nu_{11}, \ldots$ and $\alpha_{11}, \ldots$ are constants. From the energy integral

$$V_z^2 \equiv 2\eta\Phi = 2\eta(\varphi + \mathcal{E}) - V_x^2 - V_y^2,$$
$$V_x = -\tfrac{1}{2}\ \Omega y + p_x,\ V_y = \tfrac{1}{2}\Omega x + p_y, \qquad (3.7)$$

where $H_z = (c/\eta)\,\Omega$ is a uniform magnetic field, it follows that

$$d^2\Phi / dz^2 = 4\pi(\rho - \rho_0),$$
$$4\pi\eta\rho_0 = \nu_{11}^2 + \tfrac{1}{2}\ \Omega^2 + 2\ \nu_{12}\nu_{21} + \nu_{22}^2.$$

Hence, considering the expression $\rho = I/V_z$, it is not difficult to obtain (3.3), since ρ_0 can be made negative by the choice of ν_{12} and ν_{21}. In this case, the equations for the vortex:

$$\eta\frac{\partial\mathcal{E}}{\partial x} = V_y\left(\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y}\right),\ \frac{\partial\mathcal{E}}{\partial y} = -V_x\left(\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y}\right),$$

impose four conditions on the seven arbitrary constants $\nu_{11}, \alpha_{11}, \ldots$. In particular, $\nu_{11} = -\nu_{21}$, which means that the velocities V_x and V_y are solenoidal.

3.3. It is interesting to note the solution for an irrotational beam with a nonsolenoidal velocity in the xy-plane. If we let $\nu_{11} = \nu_{22}$, $\mathcal{E} = 0$, and $\nu_{11} + \nu_{22} = \nu$, it is easy to obtain

$$\rho = \frac{I}{\nu}\frac{d}{dz}(1 - e^{-\nu t}),\quad t = \int\frac{dz}{V_z},\quad I = \text{const}. \qquad (3.8)$$

from the continuity equation.

Considering (3.8), from the Poisson equation and the energy integral it is easy to derive an equation for the trajectory z(t):

$$\frac{d^2z}{dt^2} = -\omega^2 z + 4\pi\eta\frac{I}{\nu}[1 - e^{-\nu t}],$$
$$\omega^2 \equiv 2\nu_{12}^2 + \tfrac{1}{2}\Omega^2 + \nu_{11}^2 + \nu_{22}^2.$$

The solution of this equation for the case of $(\partial\Phi/\partial z)_{\Phi=0} = 0$ has the form

$$z = \frac{4\pi\eta I}{\nu(\omega^2 + \nu^2)}\left[1 - e^{-\nu t} - \frac{\nu}{\omega}\sin\omega t + \frac{\nu^2}{\omega^2}(1 - \cos\omega t)\right] \qquad (3.9)$$

Solution (3.9) has meaning up to the first point t_1

$$\exp(-\nu t_1) = \cos\omega t_1 - (\nu/\omega)\sin\omega t_1,\ \Phi_{t=t_1} = 0$$

at which a virtual cathode is formed.

§ 4. A spiral nonrelativistic beam. By separating the variables in the spiral u- and v-coordinates

$$x + iz = r_0\exp[b_1 u + b_2 v + i(b_1 v - b_2 u)],\quad r_0^2 = (b_1^2 + b_2^2)^{-1} = \text{const}, \qquad (4.1)$$

we can obtain degenerate solutions for rotational beams. This question is considered below and applied to a two-dimensional (x,z) plane nonrelativistic beam with velocity components $(\partial\chi/\partial x,\ V_y,\ \partial\chi/\partial z)$ and vorticity in

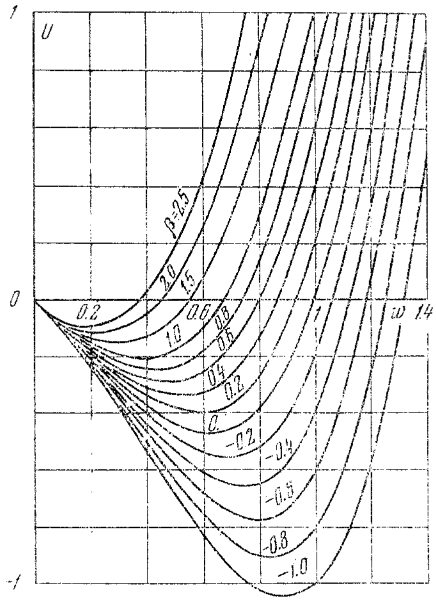

Fig. 5

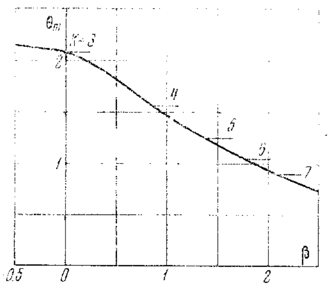

Fig. 6

the component

$$V_y = \eta c^{-1} A_y + p_y. \qquad (4.2)$$

The equations of a plane beam have the form

$$\nabla^2 A_y = 0,\ \nabla\varphi = 4\pi\rho,\ \nabla(\rho\nabla\chi) = 0,$$
$$2\eta(\varphi + \mathcal{E}) = (\nabla\chi)^2 + V_y^2,\ \eta\, d\mathcal{E}/dp_y = V_y,$$

where ∇ is defined by (1.8).

We convert in (4.2) to the u- and v-coordinates, and let

$$A = A(v),\ \mathcal{E} = \mathcal{E}(v),\ \chi = \chi(u),$$

it is easy to obtain

$$e^{-2b_1 u} w^2 = 2\eta\Phi \equiv e^{2b_2 v}\, 2\eta(\varphi - \psi),$$
$$2\eta\psi \equiv V_y^2 - 2\eta\mathcal{E},\ \frac{\eta}{c}A_y = \Omega v, \qquad (4.3)$$
$$\eta\frac{d\mathcal{E}}{dp_y} = V_y,\quad \frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial v^2} = \frac{4\pi}{w}J(v)\, e^{2b_1 u + 2b_2 v},$$
$$w \equiv d\chi/du,\quad \Omega = \text{const}.$$

If the arbitrary functions ψ and J are defined as

$$\psi = \pi b_2^{-2}\rho_0 e^{-2b_2 v} + 2b_2 v\, Q + \psi_0,\quad J = I\, e^{-4b_2 v}, \quad (4.4)$$

where ρ_0, Q, I, and ψ_0 are constants, we obtain

$$\frac{d^2\Phi}{du^2} + 4b_2{}^2\Phi = \frac{4\pi}{w} e^{2b_1u} - 4\pi\rho_0, \quad w = e^{2b_1u}\sqrt{2\eta\Phi}. \quad (4.5)$$

The cases of polar symmetry (r, y, and θ) that follow from (4.5) appear the simplest.

4.1. In the case of an azimuthal dependence

$$b_1 = 0, \quad b_2 = 1/r_0, \quad r = r_0 \exp(v/r_0), \quad \theta = u/r_0,$$

it follows from (4.5) that

$$d^2\Phi / d\theta^2 + 4\Phi = \quad (4.6)$$
$$= 4\pi I r_0^2 (2\eta\Phi)^{-1/2} - 4\pi\rho_0 r_0^2.$$

When $\rho_0 = 0$, Eq. (4.6) becomes the known equation for a beam with one velocity component ($V_y = 0$) [10]. From (4.6) follows the integral

$$w^2 (dw / d\theta)^2 = b - U, \quad U \equiv w^4 + \beta w^2 - w, \quad (4.7)$$
$$w \equiv \sqrt{2\eta\Phi} / V_0, \quad \beta \equiv \rho_0 V_0 / 2I,$$
$$V_0^3 \equiv 8\pi\eta I r_0^2.$$

Figure 5 shows the potential well U as a function of β. In the case of b = 0, we have solutions $U(w_m) = 0$, periodic in θ, for any β with amplitude w_m and wavelength

$$\theta_m = 2 \int_0^{w_m} \frac{\sqrt{w}dw}{\sqrt{1 - \beta w - w^3}}. \quad (4.8)$$

If the beam occupies the entire plane, the wave period $w(\theta)$ must be a multiple of 2π, as is arbitrarily represented by the points in Fig. 6. In particular, for $\beta = 0$ precisely one value (k = 3) is found.

In an azimuthal beam, the geometric effect (the term 4Φ in (4.6)) is stronger than the vortex effect, and (4.6) describes the periodic solutions even for negative ρ_0. The expression for rotational momentum, which follows from (4.3) and (4.4), has the form

$$p_y =$$
$$= -\Omega v + 2 b_2 \eta Q / \Omega - (\pi \eta \rho_0 / b_2 \Omega) \exp(-2b_2 v).$$

In the opposite case of a radial dependence:

$$b_2 = 0, \quad b_1 = 1 / r_0, \quad r = r_0 \exp(u / r_0), \quad \theta = u / r_0,$$

the problem reduces to the equation

$$\frac{1}{2}\left(\frac{d\Phi}{du}\right)^2 + 4\pi\rho_0\Phi = 4\pi I \frac{1}{\eta}\left(\frac{d\chi}{du} - \frac{\chi}{r_0}\right) + b, \quad (4.9)$$
$$2\eta\Phi \equiv \frac{d\chi}{du}.$$

A periodic solution cannot be constructed here for Φ, since the effect of χ increases monotonically, and there is only one point $\chi = 0$ where the virtual cathode can be placed when b = 0: $\Phi = d\Phi/dr = 0$.

4.2. It is interesting to note the solution for an irrotational plane beam in a magnetic field of the form

$$\frac{\eta}{c} A_y = A e^{-b_2 v}, \quad A = A_0 \cos b_2 u + B_0 \sin b_2 u, \quad (4.10)$$

which is similar to the solution for a rotational beam outside a magnetic field.

In fact, if we let $\mathscr{E} = p_y = 0$ and $d\chi/du = w(u)$ in (4.2) and take (4.10) into account, it is easy to obtain in spiral coordinates

$$\rho = J(v) / w, \quad e^{-2b_1u} w^2 = 2\eta\Phi, \quad \varphi = \left(\Phi + \frac{A^2}{2\eta}\right) e^{-2b_2v}, \quad (4.11)$$
$$\frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial v^2} = 4\pi\rho \exp(-2b_1u - 2b_2v).$$

Let J(v) be defined by (4.4). Then from (4.10) and (4.11) we have an equation that coincides with (4.5), where the constant ρ_0 is defined in terms of the amplitude of the magnetic potential

$$\eta 4\pi \rho_0 = b_2^2 (A_0^2 + B_0^2) \geqslant 0. \quad (4.12)$$

In particular, for an azimuthal beam ($b_1 = 0$) we can construct a solution in the form of (4.7) for positive β.

Thus, the effect of vorticity in the generalized momentum of a plane beam is equivalent to the compensating effect of a periodic magnetic field on an irrotational plane beam with the same geometry.

I thank A. N. Ievlev for assistance in the computational and graphic work.

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9 June 1967 Moscow